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Mixed convection boundary layer flow on a vertical surface in a saturated porous medium

J. H. MERKIN

Department of Applied Mathematical Studies, University of Leeds, Leeds, England

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SUMMARY

The flow of a uniform stream past an impermeable vertical surface embedded in a saturated porous medium and which is supplying heat to the porous medium at a constant rate is considered. The cases when the flow and the buoyancy forces are in the same direction and when they are in opposite direction are discussed. In the former case, the flow develops from mainly forced convection near the leading edge to mainly free convection far downstream. Series solutions are derived in both cases and a numerical solution of the equations is used to describe the flow in the intermediate region. In the latter case, the numerical solution indicates that the flow separates downstream of the leading edge and the nature of the solution near this separation point is discussed.

Nomenclature

- g acceleration due to gravity
- K permeability of the porous medium
- k thermal conductivity of the saturated porous medium
- q local heat transfer rate (constant)
- T temperature
- T_0 temperature of the ambient fluid
- T_w constant wall temperature (isothermal wall problem)
- T_s excess surface temperature (non-dimensional)
- u Darcy's law velocity in the x-direction
- U_0 free stream velocity in the x-direction
- U_s velocity on the surface (non-dimensional)
- v Darcy's law velocity in the y-direction
- x vertical co-ordinate
- y horizontal co-ordinate
- α equivalent thermal diffusivity
- β coefficient of thermal expansion
- ϵ non-dimensional parameter $= \rho_0 g\beta K(T_w - T_0)/\mu U_0$
- μ viscosity of convective fluid
- ρ_0 density of convective fluid

1. Introduction

The problem of heat transfer in flows in saturated porous media has been the subject of several recent papers. The free convection boundary-layer flow about a vertical flat surface has been treated by Cheng and Minkowycz [1], about a vertical cylinder by Minkowycz and Cheng [2], about a horizontal surface by Cheng and Chang [3]. The effects of lateral mass flux through the boundary have been discussed by Cheng [4] and Merkin [5].

In this paper we are concerned with the mixed convection boundary-layer flow about a vertical flat impermeable surface embedded in a saturated porous medium. The case when the surface is held at a constant temperature different to that of the ambient fluid has been discussed previously by Cheng [6]. In fact, he considered the wedge geometry and looked for the possible surface temperature distributions for which the governing equations have a similarity solution, of which the isothermal vertical flat surface is an example. (When the surface is horizontal the solution is essentially different to that given in [6] and has been treated by Cheng [7]). This problem was used by Cheng [6] to model flows in geothermal reservoirs where the withdrawal or re-injection of fluid can set up applied pressure gradients which cause the geothermal fluid in the reservoir to flow over impermeable surfaces.

The similarity equations derived in [6] involve the non-dimensional parameter $\epsilon = \rho_0 g \beta K(T_w - T_0)/\mu U_0$ which describes the relative importance of natural to forced convection. Now there are essentially two configurations to consider. Firstly, the buoyancy forces can be in the same direction as the flow and thus aid the flow development, or they can be in the opposite direction to the flow and so oppose it. In the aiding case we have $\epsilon > 0$ and solutions are possible for all values of ϵ and are described fully in [6]. However, for $\epsilon < 0$ we have the opposing case and a solution is possible only for a limited range of ϵ, namely $0 \geqslant \epsilon \geqslant -1.354$. Furthermore, for $-1 > \epsilon > 1.354$ the solution is non-unique and we obtain two values of the wall heat transfer for a given value of ϵ. Solutions are given in [6] only for ϵ in the range $-1 \leq \epsilon \leq 0$.

The main purpose of this paper is to consider the analogous mixed convection problem, whereby, instead of the surface being held at a constant temperature, we assume that heat is supplied to fluid at the wall at a constant rate. It is not immediately obvious in the applications discussed above whether the isothermal or constant heat flux boundary condition is the more realistic and so it would seem necessary for both cases to be considered. Unlike the isothermal wall case, a similarity solution of the governing equations is not now possible and to solve the present problem we follow the methods used by Wilks [8, 9] to solve corresponding mixed convection boundary-layer flow on a vertical plate with uniform heat flux. A series expansion in powers of x (x measures distance along the wall) is first obtained to describe the flow near the leading edge. This expansion is then extended by a numerical solution of the boundary-layer equations, which starts at $x = 0$ and in the aiding case proceeds along the wall until the asymptotic solution (i.e. for large x) is attained to the required accuracy. In the opposing case the buoyancy forces retard the fluid in the boundary layer and we would expect the flow to separate at some point x_s downstream of the leading edge. This is found to be the case and the numerical solution cannot be continued past x_s. Near x_s the solution appears to be approaching a singularity and the nature of this singularity is discussed where it is found that the (non-dimensional) surface temperature T_s behaves like $1 + A(x_s - x)^{\frac{1}{2}}$ for some constant A.

2. Equations

We consider the problem of a semi-infinite vertical impermeable flat surface embedded in a saturated porous medium over which is flowing a uniform stream U_0, and which supplies heat to the porous medium at a constant rate q. We assume that the convective fluid and the porous medium are isotropic, in thermodynamic equilibrium and have constant physical properties and that the Boussinesq approximation is valid. The flow is assumed to be described adequately by Darcy's law and if we make the further assumption of large Rayleigh number, the usual boundary-layer simplifications can be made, Wooding [10]. Here there is an outer region with velocity $u = U_0$, $v = 0$ and pressure gradient $\partial p/\partial x = -U_0\mu/K$, and next to the wall there is a boundary layer in which $\partial p/\partial y = 0$. This leads to the following boundary-layer equations,

$$u = U_0 \pm \frac{g\beta K\rho_0}{\mu}(T - T_0), \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where K is the permeability of the porous medium, α the equivalent thermal diffusivity and ρ_0 and T_0 the density and temperature (respectively) of the ambient fluid. x and y are co-ordinates measuring distance along and normal to the vertical surface respectively, and u and v are the velocities as given by Darcy's law in the x and y directions.

If we take q to be the positive throughout, then, when the flow is vertically upwards, we have the aiding case and so require the upper sign in equation (1), whereas when the flow is vertically downwards we have the opposing case and the lower sign is required. This situation would be reversed if q were negative.

The boundary conditions are

$$v = 0, \quad \frac{\partial T}{\partial y} = -\frac{q}{k} \quad \text{on } y = 0; \tag{4}$$

$$u \to U_0, \quad T \to T_0 \quad \text{as } y \to \infty. \tag{5}$$

It is appropriate at this stage to consider the similarity solution obtained by Cheng [6] for the constant wall temperature boundary condition. Here (4) is replaced by

$$v = 0, \quad T = T_w \quad \text{on } y = 0 \tag{6}$$

where, for the the sake of argument, T_w can be taken as positive. Equations (1) (2) (3) and (6) possess a similarity solution which is obtained by writing $\psi = (2\,\alpha\,U_0\,x)^{\frac{1}{2}}\,F(\eta_s)$, $T - T_0 = (T_w - T_0)G(\eta_s)$ where $\eta_s = \{U_0/(2\alpha x)\}^{\frac{1}{2}}y$, and ψ is the stream function defined from (2) in the usual way. Equation (1) gives

$$F' = 1 + \epsilon G \tag{7}$$

where the non-dimentional parameter $\epsilon = \pm g\beta K\rho_0(T_w - T_0)/(\mu U_0)$ represents the relative importance of free to forced convection with $\epsilon > 0$ in the aiding case and $\epsilon < 0$ in the opposing case. Using (7), equation (3) becomes

$$F''' + FF'' = 0 \tag{8}$$

with boundary conditions

$$F(0) = 0, \quad F'(0) = 1 + \epsilon, \quad F' \to 1 \text{ as } \eta_s \to \infty \tag{9}$$

where primes denote differentiation with respect to η_s.

For $\epsilon > 0$ solutions of equation (8) can be obtained for all ϵ, and these are given in [6]. However, for $\epsilon < 0$, (8) has solutions only in the range $-1.354 \leqslant \epsilon \leqslant 0$ and for ϵ in the range $-1.354 < \epsilon < -1$ we find that the solution is not unique, there being two solutions F_1 and F_2 for a given ϵ. This can be seen from Figure 1 where $F''(0)$ is plotted against ϵ. When $\epsilon = -1$, (9) gives $F'(0) = 0$ and we have the well-known Blasius Solution (with $F''(0) = 0.46960$). For $\epsilon < -1$ one set of solutions F_1 continues from this solution. The other set of solutions F_2 have $F_2''(0) < F_1''(0)$ for a given ϵ and are such that $F_2''(0) \to 0$ as $\epsilon \to -1$ (though there is no solution to (8) with $F'(0) = F''(0) = 0$). In Table 1 the values of $F_1''(0)$ and $F_2''(0)$ are given for ϵ in the range $-1.354 \leqslant \epsilon \leqslant -1$. So from this it would appear that the solution in [6] is relevant to the physical problem only for $\epsilon \geqslant -1$.

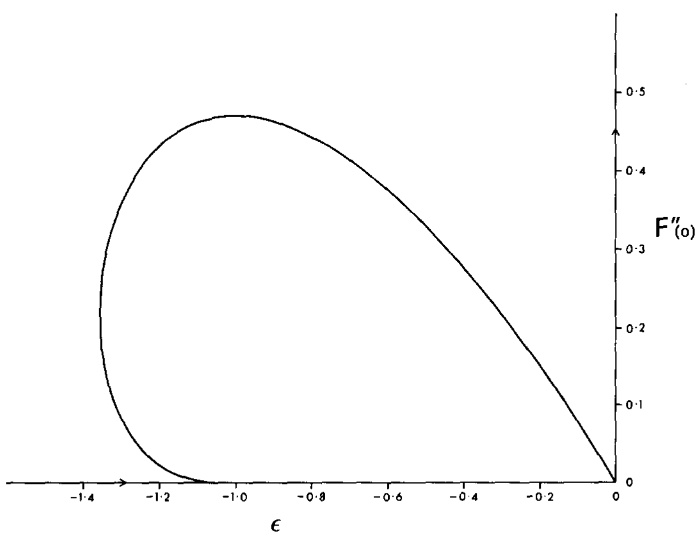

Figure 1. Graph of $F''(0)$ against ϵ for $\epsilon < 0$.

TABLE 1

The values of $F_1''(0)$ and $F_2''(0)$

ϵ	$F_1''(0)$	$F_2''(0)$
−1.00	0.46960	
−1.05	0.46758	0.00004
−1.10	0.46105	0.00194
−1.15	0.44907	0.00866
−1.20	0.43015	0.02219
−1.25	0.40152	0.04539
−1.30	0.35664	0.08497
−1.35	0.25758	0.17856
−1.354	0.22428	

3. Numerical solution

From (2) we can define a stream function ψ so that $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ then make the equations non-dimensional by writing

$$\psi = \left(\frac{U_0 \mu k}{\rho_0 g \beta K q}\right) \Psi(X,Y), \quad T - T_0 = \frac{U_0 \mu}{\rho_0 g \beta K}\, \theta(X,Y),$$

$$X = \frac{\alpha}{U_0} \left(\frac{\rho_0 g \beta k q}{\mu U_0 k}\right)^2 x \quad \text{and} \quad Y = \left(\frac{\rho_0 g K q}{\mu U_0 k}\right) y.$$

Equation (1) gives $\partial\Psi/\partial y = 1 \pm \theta$, and using this, equation (3) becomes

$$\frac{\partial^3 \Psi}{\partial Y^3} = \frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial X \partial Y} - \frac{\partial \Psi}{\partial X}\frac{\partial^2 \Psi}{\partial Y^2} \tag{10}$$

with boundary conditions

$$\Psi = 0, \quad \frac{\partial^2 \Psi}{\partial Y^2} = \mp 1 \text{ on } Y = 0, \quad \frac{\partial \Psi}{\partial Y} \to 1 \text{ as } Y \to \infty. \tag{11}$$

(Where the upper sign is taken throughout for the aiding case and the lower sign for the opposing case).

Near the leading edge there is little opportunity for heat from the wall to be taken into the fluid and so the buoyancy forces have only a small effect and the heat transfer will be predominately by forced convection. So a solution valid for small values of X is obtained by perturbing about the forced convection solution, namely $\Psi = Y$. When this is done it turns out that it is more natural to use $\eta = \frac{1}{2} Y/X^{\frac{1}{2}}$ rather than Y as an independent variable. This suggests writing $\Psi = Y \pm 4Xf(X, \eta)$ to put (10) into a form more appropriate for solving from $X = 0$. Equation (10) then becomes

$$\frac{\partial^3 f}{\partial\eta^3} + 2\eta\frac{\partial^2 f}{\partial\eta^2} - 2\frac{\partial f}{\partial\eta} = \pm 4X^{\frac{1}{2}}\left\{\left(\frac{\partial f}{\partial\eta}\right)^2 - 2f\frac{\partial^2 f}{\partial\eta^2}\right\} \tag{12}$$

$$+ 4X\frac{\partial^2 f}{\partial\eta\partial X} \pm 8X^{\frac{3}{2}}\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\partial X} - \frac{\partial f}{\partial X}\frac{\partial^2 f}{\partial\eta^2}\right)$$

with boundary conditions

$$f = 0, \quad \frac{\partial^2 f}{\partial\eta^2} = -1 \text{ on } \eta = 0, \quad \frac{\partial f}{\partial\eta} \to 0 \text{ as } \eta \to \infty. \tag{13}$$

$f(X,\eta)$ can be expanded in the form

$$f(X,\eta) = f_0(\eta) \pm X^{\frac{1}{2}} f_1(\eta) + Xf_2(\eta) + \ldots \tag{14}$$

where

$$f_0(\eta) = \tfrac{1}{4}(1 - erfc\eta(1 + 2\eta^2)) - \frac{1}{2\sqrt{\pi}}\eta e^{-\eta^2}, \tag{15}$$

The higher order terms can then be found in a straightforward, though laborous, way. We can define a non-dimensional excess surface temperature T_s by $T_s = (\rho_0 g\beta K)/(\mu U_0)(T - T_0)$ so that $T_s = \theta(X,0) = 2X^{\frac{1}{2}}(\partial f/\partial\eta)_0$. Note that the non-dimensional streamwise velocity component on the surface U_s is from (1) given by $U_s = 1 \pm T_s$. (14) then gives, for small X,

$$T_s = X^{\frac{1}{2}}(1.12838 \pm 0.38662\,X^{\frac{1}{2}} + 0.33872\,X \pm \ldots). \tag{16}$$

TABLE 2

Excess surface remperature T_s – opposing case

X	Numerical solution	Series (from 16))
0	0.00000	0.00000
0.04	0.24462	0.24385
0.09	0.38752	0.38246
0.16	0.55654	0.53489
0.25	0.79220	0.70316
0.263	0.83615	
0.2756	0.89184	
0.2822	0.93089	
0.28556	0.95996	
0.28723	0.98694	
0.28744	0.99687	
0.28745	1.00000	

The solution given by (14) describes the situation accurately only for small X and to obtain the solution for larger values of X equation (12) was solved numerically. The method used was similar to that used by the author [5] and to that described by Wilks [9]. Two integrations were performed in each case, with step lengths h in the η direction of 0.1 and 0.05 and then Richardson's h^2-extrapolation formula (Smith [11]) was used to improve the accuracy of the results. The numerical integration started at $X = 0$, where f can be found from (15), and then proceeded in a stepwise manner. In the opposing case the numerical solution terminated at the point $X_s = 0.28745$ and could not be continued past this point, for the numerical solution shows that, as $X \to X_s$, $T_s \to 1$ and $dT_s/dx \to \infty$. In this case $U_s = 1 - T_s$ so that as $T_s \to 1$, $U_s \to 0$ and at $X = X_s$ a flow reversal is indicated with the boundary layer leaving the surface there. Values of T_s are given in Table 2 together with values of T_s as calculated from (16), and it can be seen that there is good agreement only up to $X = 0.16$, (16) being in error by about 4% there.

TABLE 3

Excess surface temperature T_s – aiding case

X	Numerical solution	Series
0	0.0000	0.0000
0.09	0.3106	0.3129
0.16	0.4046	0.4112
0.25	0.4950	0.5099
0.36	0.5824	0.6110
0.49	0.6671	0.7166
0.64	0.7494	
0.81	0.8295	
1.00	0.9076	
1.40	1.0464	
1.80	1.1622	
2.20	1.2628	
2.60	1.3525	
3.00	1.4338	
3.80	1.5778	
4.60	1.7036	
5.20	1.7890	
6.00	1.8936	1.9475
8.00	2.1207	2.1667
12.0	2.4828	2.5195
18.0	2.9001	2.9294
25.0	3.2842	3.3084
41.0	3.9513	3.9695
66.0	4.7093	4.7231
89.0	5.2522	5.2637
121	5.8683	5.8800
185	6.8377	6.8451
249	7.6020	7.6081
377	8.8052	8.8099
633	10.5644	10.5678
889	11.8947	11.8974
1401	13.9307	13.9326

In the aiding case no such difficulty is encountered. However, for large X, equation (12) is no longer the appropriate equation to solve and a further transformation of (10) is required. Now far downstream of the leading edge free convection will predominate and this suggests putting

$$\Psi = 4X^{\frac{2}{3}}g(X,\zeta), \quad \text{where } \zeta = Y/2X^{\frac{1}{3}}.$$

Equation (10) then becomes

$$\frac{\partial^3 g}{\partial \zeta^3} + \frac{16}{3}\, g\, \frac{\partial^2 g}{\partial \zeta^2} - \frac{8}{3}\left(\frac{\partial g}{\partial \zeta}\right)^2 = 8X\left[\frac{\partial g}{\partial \zeta}\,\frac{\partial^2 g}{\partial \zeta \partial X} - \frac{\partial g}{\partial X}\,\frac{\partial^2 g}{\partial \zeta^2}\right] \tag{17}$$

with boundary conditions

$$g = 0, \quad \frac{\partial^2 g}{\partial \zeta^2} = -1 \text{ on } \zeta = 0, \quad \frac{\partial g}{\partial \zeta} \to \frac{1}{2X^{\frac{1}{3}}} \text{ as } \zeta \to \infty. \tag{18}$$

Equation (17) was then integrated numerically by the same method that was used for (12). This integration started at $X = 1$ with the solution as obtained from a numerical integration of (12) from $X = 0$ to $X = 1$ as starting values and proceeded downstream until the asymptotic values were attained to the required accuracy. Values of T_s thus obtained are given in Table 3 together with values of T_s as obtained from (16) and from the asymptotic expansion (as described in the next section). It can be seen that (16) is in good agreement with the numerical solution only up to about $X = 0.25$.

4. Solution for large X – aiding case

To obtain a solution valid for large X, we start with equation (17). (18) suggests expanding $g(X,\zeta)$ in the form

$$g(X,\zeta) = g_0(\zeta) + X^{-\frac{1}{3}}g_1(\zeta) + X^{-\frac{2}{3}}g_2(\zeta) + X^{-1}g_3(\zeta) + \ldots \tag{19}$$

The equation for $g_0(\zeta)$ is

$$g_0''' + \tfrac{16}{3}g_0 g_0'' - \tfrac{8}{3}g_0'^2 = 0 \tag{20}$$

with boundary conditions

$$g_0(0) = 0, \quad g_0''(0) = -1 \quad g_0 \to 0 \text{ as } \zeta \to \infty. \tag{21}$$

(primes denote differentiation with respect to ζ). Equation (20) is the constant heat flux free convection problem analogous to the isothermal wall problem solved by Cheng and Minkowycz [1].

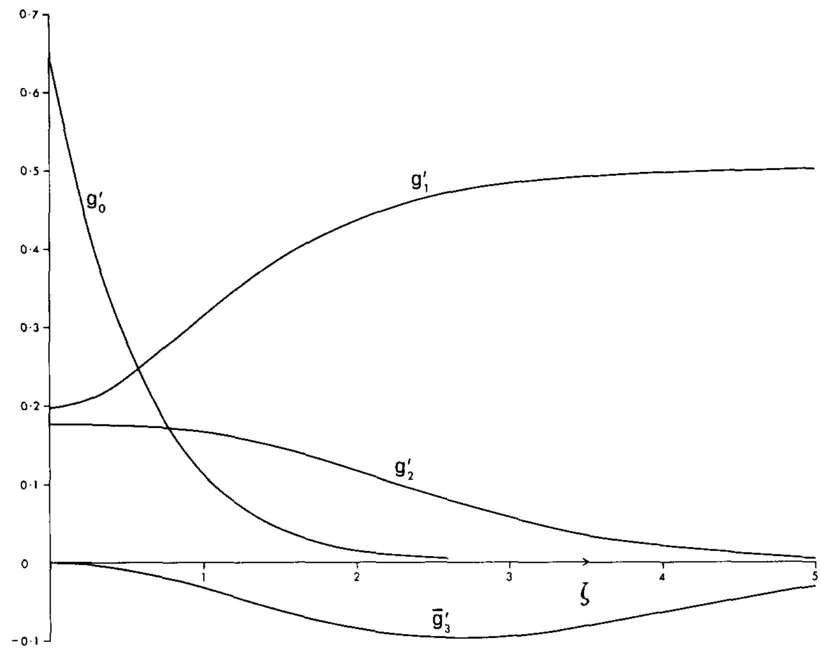

Figure 2. Graphs of g_0', g_1', g_2' and $\bar{g}_3'$.

A numerical integration of (20) gives $g_0'(0) = 0.64809$ and a graph of g_0' is given in Figure 2. The equations for g_1 and g_2 are linear and can be solved in a straightforward way once g_0 is known. We find that $g_1'(0) = 0.19864$ and $g_2'(0) = 0.17707$ and graphs of g_1' and g_2' are also given in Figure 2.

We expect indeterminancy in expansion (19), arising from the asymptotic nature of the solution in the sense described by Stewartson [12].

This appears first when we come to consider g_3, the term of $O(X^{-1})$ in the expansion. The equation for g_3 can be integrated once and using the relevant boundary condition we obtain

$$g_3'' + \tfrac{16}{3} g_0 g_3' - \tfrac{8}{3} g_0' g_3 = -\tfrac{8}{3} g_1 g_2' \tag{22}$$

with boundary conditions

$$g_3(0) = 0 \quad g_3' \to 0 \text{ as } \zeta \to \infty. \tag{23}$$

Now, for large ζ (22) becomes, on neglecting exponentially small terms

$$g_3'' + \tfrac{16}{3} C_0 g_3' = 0 \tag{24}$$

where $C_0 = \lim_{\zeta \to \infty} g_0(\zeta) = 0.37048$, and so any solution of (22) has $g_3' \to 0$ as $\zeta \to \infty$. To integrate (22) numerically we require $g_3'(0)$ and we choose that solution $\bar{g}_3$ which has $\bar{g}_3'(0) = 0$ then from the above $\bar{g}_3' \to 0$ as $\zeta \to \infty$ and so satisfies (23) A graph of $\bar{g}_3'$ is given in Figure 2.

But (22) also possesses the complementary function $g_a = \zeta g_0' - 2g_0$ which also satisfies (23) and so the full solution of (22) is $g_3 = \overline{g}_3 + \lambda g_a$ where λ is a constant that cannot be determined from the asymptotic expansion. It is interesting to note that the term of $O(X^{-1})$ can be determined without the inclusion of a term of $O(X^{-1} \log X)$ as was found necessary by Stewartson [12].

(19) gives, for large X,

$$T_s = 1.29618X^{\frac{1}{3}}(1 - 0.07924X^{-\frac{1}{3}} + 0.27321X^{-\frac{2}{3}} - \lambda X^{-1} + \ldots). \tag{25}$$

Values of T_s calculated from the first three terms in (25) are given in Table 3, from which it can be seen that the error is about 1% at $X = 18$ and about 0.05% at $X = 377$. By comparing the values of T_s given by the numerical integration with those given by (25), an estimate for λ can be obtained. We find $\lambda = 0.19$.

5. Solution near the singularity – opposing case

When the flow is vertically downwards the buoyancy forces act in a direction opposite to that of the flow. The fluid in the boundary layer is thus retarded and we expect the boundary layer to separate from the surface at some point downstream of the leading edge. This is indicated by the numerical solution terminating at $X_s = 0.28745$ with $T_s \to 1$, $U_s \to 0$ in a singular way. From a closer examination of the results given in Table 2 it appears that U_s behaves like $(X_s - X)^{\frac{1}{2}}$ near $X = X_s$. This is suggested by a log-log plot of U_s against $X_s - X$, as shown in Figure 3, where the values of U_s close to X_s appear to lie on a straight line of slope approximately 0.5. Actually the slope of the line is 0.53 which is about as close to the required value as could be expected since difficulties in finding the correct value of the exponent for the behaviour of the solution near separation from plots of this kind have been reported previously by Wilks [9] and Buckmaster [13].

We now discuss in more detail the behaviour of the solution near $X = X_s$ in a manner similar to that suggested originally by Goldstein [14] for the behaviour of an incompressible boundary layer near separation. It is interesting to note that though, as in [14], a $(X_s - X)^{\frac{1}{2}}$ behaviour appears, it comes from a different form of expansion to that given in [14]. We first transform equation (10) by putting $\xi = X_s - X$, $\Psi = (3\xi/2)^{\frac{2}{3}}\phi(\xi,\tau)$ where $\tau = y/(\frac{3}{2}\xi)^{\frac{1}{3}}$. This transformation is the only one compatible with (10) and the boundary condition on $y = 0$ as given by (11). Equation (10) becomes

$$\frac{\partial^3\phi}{\partial\tau^3} + \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \phi\frac{\partial^2\phi}{\partial\tau^2} = \frac{3\xi}{2}\left[\frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial\phi}{\partial\tau}\frac{\partial^2\phi}{\partial\xi\partial\tau}\right] \tag{26}$$

with boundary conditions

$$\phi = 0, \quad \frac{\partial^2\phi}{\partial\tau^2} = 1 \text{ on } \tau = 0. \tag{27}$$

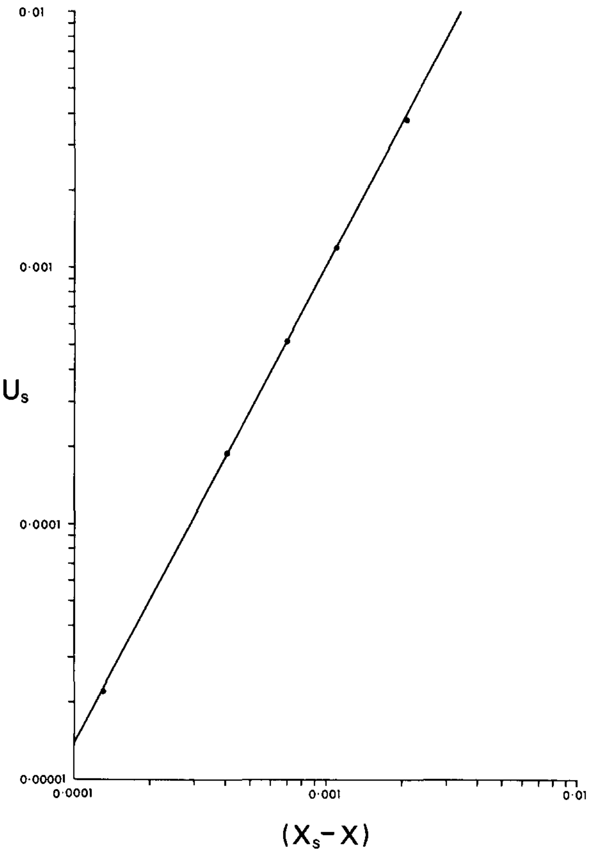

Figure 3. log-log plot of U_S against $(X_S - X)$.

As in [14] we relax the outer boundary condition, requiring only that the solution should not be exponentially large as $\tau \to \infty$. We now expand $\phi(\xi, \tau)$ in powers of $\xi^{\frac{1}{6}}$, namely

$$\phi(\xi,\tau) = \phi_0(\tau) + \xi^{\frac{1}{6}}\phi_1(\tau) + \xi^{\frac{1}{3}}\phi_2(\tau) + \xi^{\frac{1}{2}}\phi_3(\tau) + \ldots \tag{28}$$

On solving the equations which result from the substitution of (28) into (26) and equating like powers of ξ, we find

$$\phi_0(\tau) = \tfrac{1}{2}\tau^2, \tag{29}$$

$$\phi_1(\tau) = A_1\tau, \tag{30}$$

$$\phi_2(\tau) = A_2\tau - A_1^2\,\frac{\tau^3}{3} \tag{31}$$

for some constants A_1 and A_2. The equation for ϕ_3 is

$$\phi_3''' - \tfrac{1}{2}\tau^2\phi_3'' + \tfrac{7}{4}\tau\phi_3' - \tfrac{7}{4}\phi_3 = -\tfrac{9}{16}A_1{}^3\tau^2 - \tfrac{7}{4}A_1A_2 \tag{32}$$

where primes denote differentiation with respect to τ. The solution of (32), not exponentially large at infinity, which has $\phi_3(0) = 0$ is

$$\phi_3(\tau) = A_3\tau - \tfrac{3}{4}A_1^3\tau^2 - A_1A_2\tau \int_0^\tau \left[\frac{1}{s^2} + \frac{2\sqrt{\pi}}{6^{\frac{2}{3}}\Gamma(2/3)} U\left(-\frac{1}{2}; \frac{5}{3}; \frac{s^3}{6}\right)\right] ds \tag{33}$$

where $U(-\frac{1}{2}; \frac{5}{3}; s^3/6)$ is the confluent hypergeometric function not exponentially large at infinity and, from Slater [15],

$$U\left(-\frac{1}{2}; \frac{5}{3}; \frac{s^3}{6}\right) = \frac{-\Gamma(2/3)6^{\frac{2}{3}}}{2\sqrt{\pi}s^2} + \frac{\Gamma(-2/3)}{\Gamma(-7/6)} + O(s)$$

for small s and so the integral in (33) is bounded as $\tau \to 0$.

Applying the final boundary condition $\phi_3''(0) = 0$ determines A_2 in terms of A_1, namely $A_2 = \frac{18}{7} 3^{\frac{2}{3}}\{\Gamma(2/3)/\Gamma(1/3)\}^2 A_1^2$ (on simplifying the gamma functions). A_3 is a constant which cannot be determined at this stage, but which is found in terms of A_1 when the term of $O(\xi^{\frac{2}{3}})$ is considered. The process is the same as above, and omitting all the details we find

$$A_3 = \frac{9}{28}\, 12^{\frac{1}{3}} \frac{\Gamma(2/3)^4}{\Gamma(1/3)^4} \left(\frac{39}{2} - \frac{108}{7} 2^{\frac{1}{3}}\right) A_1^3,$$

so upto this stage the solution near $\xi = 0$ involves only the one arbitrary constant A_1, though other terms may appear in the expansion as in the modification of [14] by Stewartson [16], but in terms of higher order than those considered here.

Finally we find that near $X = X_s$ ($\xi = 0$)

$$T_s = 1 - (\tfrac{3}{2})^{\frac{1}{3}}\xi^{\frac{1}{2}}(A_1 + 1.3575A_1^2\xi^{\frac{1}{6}} + 0.0029A_1^3\xi^{\frac{1}{3}} + \dots) \tag{34}$$

and the $(X_s - X)^{\frac{1}{2}}$ behaviour is recovered. An estimate for A_1 can be found from Figure 3, and we find that $A_1 \simeq 1.02$.

6. Conclusion

In order to test the applicability of the above analysis to the geothermal problem we use the same data as Cheng [6], namely $\beta = 1.8 \times 10^{-4}/°K$, $\rho_0 = 10^3$ kgm/m^3, $K = 10^{-12}$ m^2, $\mu = 2.7 \times 10^{-4}$ kmg/ms, $\alpha = 6.3 \times 10^{-7}$ m^2/s. Then with a flow speed $U_0 = 0.1$ cm/hour (the mid value of the range suggested in [6]) we find that thr critical value of $\epsilon = -1$ will result from an applied temperature difference of about 43 °K between the impermeable surface and the ambient fluid.

For the constant heat flux problem we can determine the heating rate q which would produce separation at 100m from the leading edge with the above flow speed. Taking k = 2.4W/°Km we find q = 3.6W/m. This heating rate would, in the purely free convection problem, as given by (20), produce a temperature difference of about 35°K at 100m from the leading edge.

The surface temperatures found in the above calculations are well within the range expected in this particular application. This suggests that careful consideration needs to be given in the geothermal context to the problem of flow separation which will have the result of spreading the heat supplied from the impermeable surface into a much wider region of the porous medium than the boundary layer on the surface.

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